# Interior Schauder estimate

Thm 1) (Fundamental Sch. est.)

$u \in C^{2,\alpha}(\mathbb{R}^n)$, $(\alpha \in (0,1))$, then

$$[D^2 u]_\alpha \le C [\Delta u]_\alpha$$

holds for some $C = C(n,\alpha)$.

Cor) Let $a_{ij}^\circ = A$ be a (constant) symmetric matrix w/

$$\lambda |\xi|^2 \le a_{ij}^\circ \xi_i \xi_j \le \Lambda |\xi|^2$$

$$0 < \lambda \le \Lambda,$$

Then $\exists\, C = C(n, \alpha, \lambda, \Lambda)$ s.t.

$$[D^2 u]_{\alpha;\mathbb{R}^n} \le C [a_{ij}^\circ u_{ij}]_{\alpha;\mathbb{R}^n}$$

holds for any $u \in C^{2,\alpha}(\mathbb{R}^n)$.

$(\alpha \in (0,1))$

pf) $A = VDV^{-1}$.

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. $V = [v_1 \cdots v_n]$

$Av_i = \lambda_i v_i$, $\|v_i\| = 1$, $v_i \perp v_j$ if $i \neq j$.

$\lambda |\xi|^2 \le a_{ij}\xi_i \xi_j \le \Lambda |\xi|^2$

$\iff \lambda \le \lambda_1, \dots, \lambda_n \le \Lambda$.

$V^{-1} = V^t$, $(V^{-1})^t = (V^t)^t = V$.

We define $T = D^{1/2} V^{-1}$.

where $D^{1/2} = \begin{pmatrix} \lambda_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{1/2} \end{pmatrix}$.

$\Rightarrow T^t T = (V^{-1})^t (D^{1/2})^t D V^{-1} = VDV^{-1} = A$.

$\iff T_i^k T_i^l = a_{kl}$

Define $\hat{u}(x) = u(Tx) \in C^{2,\alpha}(\mathbb{R}^n)$

$\Rightarrow \hat{u}_i = u_k T^k_i \quad , \hat{u}_{ij} = u_{kl} T^k_i T^l_j$

$\Rightarrow \Delta \hat{u}(x) = \mathring{a}_{kl} u_{kl}(Tx)$

$$\frac{|\mathring{a}_{ij} u_{ij}(x) - \mathring{a}_{ij} u_{ij}(y)|}{\|x-y\|^\alpha} = \frac{|\Delta\hat{u}(\vec{T}x) - \Delta\hat{u}(\vec{T}y)|}{\|x-y\|^\alpha}$$

$$\geq C(\lambda,\Lambda) \frac{|\Delta\hat{u}(\vec{T}x) - \Delta\hat{u}(\vec{T}y)|}{\|\vec{T}x - \vec{T}y\|^\alpha}$$

$$\Rightarrow \frac{|\Delta\hat{u}(x) - \Delta\hat{u}(y)|}{\|x-y\|^\alpha} \leq C[\mathring{a}_{ij} u_{ij}]_\alpha$$

$\Rightarrow [\Delta\hat{u}]_\alpha \leq C[\mathring{a}_{ij} u_{ij}]_\alpha$

By Thm 2). $[D^2\hat{u}]_\alpha \leq C[\mathring{a}_{ij} u_{ij}]_\alpha$

$[D^2 u]_\alpha \leq C[D^2\hat{u}]_\alpha$ as above

$\therefore [D^2 u]_\alpha \leq C(n,\alpha,\lambda,\Lambda)[\mathring{a}_{ij} u_{ij}]_\alpha$ ☒

Lemma 3) $a_{ij}, b_i, c \in C^\alpha(B_2)$

$a_{ij} = a_{ji}$, $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j$

$\|a_{ij}\|_{C^\alpha(\overline{B}_2)}, \|b_i\|_{C^\alpha}, \|c\|_{C^\alpha} \le \Lambda$.

$f = Lu = a_{ij}u_{ij} + b_i u_i + cu$

$$\Rightarrow \|u\|_{C^{2,\alpha}(\overline{B}_1)} \le C\Big(\|f\|_{C^\alpha(\overline{B}_2)} + \cancel{\sup_{B_2}|u|}\Big)$$

$\Rightarrow \|u\|_{C^2(B_2)}$

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where $C = C(n, \alpha, \lambda, \Lambda)$

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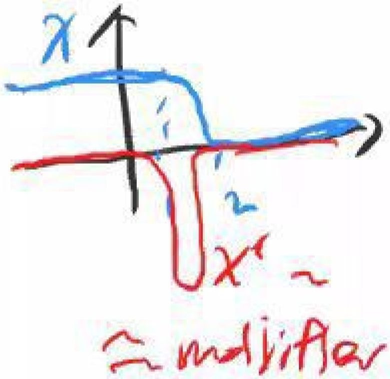
pf) We choose $\chi \in C^{2,\alpha}(\mathbb{R})$ s.t.




$\chi(r) = 1$ if $r \le 1$

$\chi(r) = 0$ if $r \ge 2$.

$\|\chi\|_{C^{2,\alpha}(\mathbb{R})} \le 10$.

Given $x_0$ w/ $B_{2\rho}(x_0) \subset B_L$ $(\rho \le 1)$,

we define $v(x) = u(x)\,\chi(\|x - x_0\|/\rho)$.

$\Rightarrow v = u$ in $B_\rho(x_0)$, $v = 0$ in $\mathbb{R}^n \setminus B_{2\rho}(x_0)$.

$v \in C^{2,\alpha}(\mathbb{R}^n)$.

Let $a_{ij}^0 = a_{ij}(x_0)$.

By cor, $[D^2 v]_{\alpha;\mathbb{R}^n} \le C_1 [a_{ij}^0 v_{ij}]_{\alpha;B_{2\rho}}$

$[a_{ij}^0 v_{ij}]_\alpha \le [(a_{ij}^0 - a_{ij}) v_{ij}]_\alpha + [a_{ij} v_{ij}]_\alpha$.

$[(a_{ij}^0 - a_{ij}) v_{ij}]_\alpha$ $\quad ([f]_0 = \sup |f|)$

$\le \sum_{i,j} [a_{ij}^0 - a_{ij}]_\alpha [v_{ij}]_0 + |a_{ij}^0 - a_{ij}|_0 [v_{ij}]_\alpha$.

$\le n^2 \left( \Lambda [v]_{2;B_{2\rho}} + \underline{(2\rho)^\alpha \Lambda} [D^2 v]_{\alpha;B_{2\rho}(x_0)} \right)$

$\left( \because |a_{ij}^0 - a_{ij}(x)| = \|x_0 - x\|^\alpha \dfrac{|a_{ij}(x_0) - a_{ij}(x)|}{\|x_0 - x\|^\alpha} \le \Lambda (2\rho)^\alpha \right)$

$$\Rightarrow [D^2 v]_{\alpha;\mathbb{R}^n} \le C_1 2^\alpha n^2 \Lambda \rho^\alpha [D^2 v]_{\alpha; B_{2\rho}(x_0)} + C_1 [a_{ij} v_{ij}]_{\alpha; B_{2\rho}} + C \|v\|_{C^2(B_{2\rho})}$$

By choosing $\rho$ small enough to have $C_1 2^\alpha n^2 \Lambda \rho^\alpha \le 1/2$, we have

$$\frac{1}{2} [D^2 v]_{\alpha;\mathbb{R}^n} \le C_1 [a_{ij} v_{ij}]_{\alpha; B_{2\rho}} + C \|v\|_{C^2(B_{2\rho})}$$

$$v = u\chi \Rightarrow v_i = u_i \chi + u \chi_i$$

$$v_{ij} = u_{ij}\chi + u_i \chi_j + u_j \chi_i + u \chi_{ij}$$

$$[a_{ij} v_{ij}]_\alpha \le [a_{ij} u_{ij} \chi]_\alpha + 2 [a_{ij} u_i \chi_j]_\alpha + [u\, a_{ij} \chi_{ij}]_\alpha$$

By using $[fg]_\alpha \le [f]_\alpha [g]_0 + [f]_0 [g]_\alpha$

$$[fgh]_\alpha \le [f]_\alpha [g]_0 [h]_0 + [f]_0 [g]_\alpha [h]_0 + [f]_0 [g]_0 [h]_\alpha$$

$$[f]_{\alpha; B_{2\rho}} \le (4\rho)^{1-\alpha} [f]_{1; B_{2\rho}}$$

& ($\because |f(x) - f(y)| = |\int_y^x f' \, ds| \le |x-y| [f]_1 \le C|x-y|^\alpha [f]_1$)

$$[a_{ij} v_{ij}]_\alpha \leq [a_{ij} u_{ij}]_\alpha [\chi]_0 + [a_{ij} u_{ij}]_0 [\chi]_\alpha$$

$$+ C \|u\|_{C^2(B_{2\rho}(x_0))}$$

$$\leq 10 [a_{ij} u_{ij}]_\alpha + C \|u\|_{C^2(B_{2\rho}(x_0))}$$

$v_{ij} = u_{ij}$ in $B_\rho(x_0)$

$$\Rightarrow [D^2 v]_{\alpha; \mathbb{R}^n} \geq [D^2 v]_{\alpha; B_\rho(x_0)} = [D^2 u]_{\alpha; B_\rho(x_0)}$$

$$\|v\|_{C^2(B_{2\rho}(x_0))} \leq C \|u\|_{C^2(B_{2\rho}(x_0))}$$

$$\therefore [D^2 u]_{\alpha; B_\rho(x_0)} \leq C \Big( [a_{ij} u_{ij}]_{\alpha; B_{2\rho}(x_0)} + \|u\|_{C^2(B_{2\rho}(x_0))} \Big)$$

$$f = Lu = a_{ij} u_{ij} + b_i u_i + cu$$

$$a_{ij} u_{ij} = f - b_i u_i - cu$$

$$\Rightarrow [a_{ij} u_{ij}]_\alpha \leq [f]_\alpha + [b_i u_i]_\alpha + [cu]_\alpha$$

$$\leq \|f\|_{C^\alpha} + C \|u\|_{C^2(B_{2\rho}(x_0))}$$

$$\therefore [D^2 u]_{\alpha; B_\rho(x_0)} \leq C \left( \|f\|_{C^\alpha(B_2)} + \|u\|_{C^2(B_2)} \right)$$

By choosing $\rho \leq 1/10$, we completes the proof!!

Lemma 3)

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$$\|u\|_{C^{2,\alpha}(B_1)} \le C\left(\|f\|_{C^{\alpha}(B_2)} + \cancel{\sup_{B_2}|u|}\right)$$

↳ Interior Schauder

$$\|u\|_{C^{2,\alpha}(B_1)} \le C\left(\|f\|_{C^{\alpha}(B_2)} + \|u\|_{C^{2}(B_2)}\right)$$

Lemma 4)

$$\|u\|_{C^{2,\alpha}(B_1)} \le C_1\left(\|f\|_{C^{\alpha}(B_2)} + [u]_{0;B_2} + [u]_{2;B_2}\right)$$

where $C_1 = C_1(n, \alpha, \lambda, \Lambda)$

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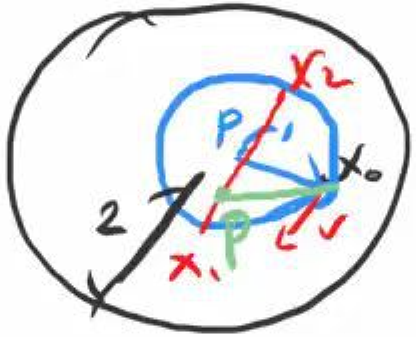
pf) It is enough to show that

$$[u]_{1;B_2} \le C\left([u]_{0;B_2} + [u]_{2;B_2}\right)$$

Interpolation inequality.

holds for a numeric constant $C$.




Given $x_0 \in B_2$, we choose a ball $B_1(p)$ such that $\overline{B_1(p)} \subset B_2(0)$, $x_0 \in \partial B_1(p)$.

Let $u_v(x_0) = \|\nabla u(x_0)\|$. $(\|v\| = 1)$

$$x_1 = p + v, \quad x_2 = p - v$$

Define $f(t) = u(p + tv) \Rightarrow f' = u_v(p + tv)$

$f \in C^2 \Rightarrow$ By the mean value theorem

$\exists \bar{t} \in (-1, 1)$ s.t. $f'(\bar{t}) = \frac{1}{2}(f(1) - f(-1))$

$\Leftrightarrow u_v(\bar{p}) = \frac{1}{2}(u(x_1) - u(x_2))$

(where $\bar{p} = p + \bar{t}v$)

$\Rightarrow |u_v(\bar{p})| \le \frac{1}{2}(|u(x_1)| + |u(x_2)|) \le [u]_{0;B_2}$

Denote. $\|x_0 - \bar{p}\| = r \le 2$. $w = \frac{x_0 - \bar{p}}{r}$

Define $g(t) = u_v(\bar{p} + tw)$

$\Rightarrow g' = u_{vw}(\bar{p} + tw)$

where $u_{vw} = w^T \nabla^2 u \, v$

So. $|g'| = |u_{vw}(\bar{p} + tw)| \leq [u]_{2;B_2}$

$$g(r) = g(0) + \int_0^r g' = u_v(\bar{p}) + \int_0^r g'$$

$$\overset{\|}{u_v(x_0)} = \|\nabla u(x_0)\|$$

$\Rightarrow \|\nabla u(x_0)\| \leq |u_v(\bar{p})| + r|g'|$

$\leq [u]_{0;B_2} + 2[u]_{2;B_2}$

$\Rightarrow [u]_{1;B_2} \leq [u]_{0;B_2} + 2[u]_{2;B_2}$

$\boxtimes$

Thm) Interpolation

$j+\beta < k+\alpha$.

Let $\Omega$ be open, $j, k \in \{0, 1, 2, \cdots\}$,

$0 \le \alpha, \beta \le 1$. Then, given $\varepsilon > 0$

and $u \in C^{k,\alpha}(\Omega)$

$$[u]_{j,\beta;\Omega} \le \varepsilon [u]_{k,\alpha;\Omega} + C[u]_{0;\Omega}$$

$$\|u\|_{C^{j,\beta}(\Omega)} \le \varepsilon \|u\|_{C^{k,\alpha}(\Omega)} + C \sup_{\Omega} |u|$$

holds for some constant $C = C(\varepsilon, k, j)$

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pf) See [GT], Section 6.8.

Lemma 5) $a_{ij}, b_i, c, f, u \in C^{2,\alpha}(B_{2\rho}(0))$

$\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha}, \|c\|_{C^\alpha} \le \Lambda$

$a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2$. $a_{ij} = a_{ji}$

$f = Lu$. , $(\rho \le 1)$

Then, $\rho^{2+\alpha}[u]_{2,\alpha;B_\rho}$

$$\le C_1\left(\rho^2[f]_{0;B_{2\rho}} + \rho^{2+\alpha}[f]_{\alpha;B_{2\rho}} + [u]_{0;B_{2\rho}} + \rho^2[u]_{2;B_{2\rho}}\right)$$

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proof) $\hat{u}(x) = u(\rho x) \Rightarrow \hat{u} \in C^{2,\alpha}(B_2(0))$

Similarly, we define $\hat{a}_{ij}, \hat{b}_i, \hat{c}, \hat{f}$

(e.g. $\hat{f}(x) = f(\rho x)$)

Then $\nabla\hat{u} = \rho\nabla u$, $\nabla^2\hat{u} = \rho^2\nabla^2 u$

$[\hat{u}]_{2,\alpha} = \rho^{2+\alpha}[u]_{2,\alpha}$

$$\hat{a}_{ij}\hat{u}_{ij} + \rho\hat{b}_i\hat{u}_i + \rho^2\hat{c}\hat{u} = \rho^2\hat{f}$$

$$[\hat{a}_{ij}]_0 = [a_{ij}]_0 \leq \Lambda$$

$$[\hat{a}_{ij}]_\alpha = \rho^\alpha [a_{ij}]_\alpha \leq 1^\alpha \cdot \Lambda = \Lambda$$

$$[\rho\hat{b}_i]_0 \leq 1 \cdot \Lambda \cdot \text{---}$$

$$\|\hat{a}_{ij}\|_{C^\alpha}, \|\rho\hat{b}_i\|_{C^\alpha}, \|\rho^2\hat{c}\| \leq \Lambda$$

By Lemma 4.

$$[\hat{u}]_{2,\alpha;B_1} \leq C_1\Big(\rho^2\|\hat{f}\|_{C^\alpha(B_2)} + [\hat{u}]_{0;B_2} + [\hat{u}]_{2;B_2}\Big)$$

$$\Rightarrow \rho^{2+\alpha}[u]_{2,\alpha;B_\rho}$$

$$\leq C_1\Big(\rho^2[f]_{\alpha;B_{2\rho}} + \rho^{2+\alpha}[f]_{\alpha;B_{2\rho}} + [u]_{0;B_{2\rho}} + \rho^2[u]_{2;B_{2\rho}}\Big)$$